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Swirling flow through a convergent funnel

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The work that follows considers the velocity profiles within the boundary layer at the wall of an arbitrarily converging funnel. The occurrence of super-velocities, i.e. components of velocity within the boundary layer exceeding their corresponding free stream component, is investigated and the relevance of such a phenomenon to the efficiency of discharge discussed.

1. Introduction

Swirling flow in a perfectly conical funnel was first discussed by Taylor (1950) when the flow was assumed to have a component of velocity as given by a potential line vortex lying along the axis of the cone—an idealization of an agricultural sprayer. It was shown that the radial pressure gradient which necessarily accompanies the swirling motion acts on the boundary layer driving it along the surface of the cone towards the apex. Within the boundary layer fluid is retarded by viscosity and consequently has not sufficient centrifugal acceleration to hold it in a circular path against the inward radial pressure gradient.

The natural extension of the problem to include radial as well as swirling flow in the mainstream attracted the attention of Binnie & Harris (1950) who, using an extension of Taylor's Pohlhausen approach, dealt with swirling mainstream flow through a convergent-divergent funnel made up of sections of perfect cones of differing semi-angles $< 10°$. A more ideal mathematical approach, however, was made by Garbsch (1955, 1956), who considered the flow in a perfect cone as a super-imposition of swirl and the flow due to a sink placed at the apex. Both the pieces of work mentioned in this paragraph conclude that such a flow is exceptional in that it is a case in which viscosity has a favourable effect upon the efficiency of discharge.

Little has been said as to the shape of the velocity profiles occurring in the boundary layer at the wall when both components of velocity exist in the free stream. Will the swirl effect within the boundary layer cause the velocity towards the apex to exceed the free stream component in that direction (a phenomenon which shall henceforward be referred to as 'super-velocity')? That such a phenomenon might exist poses an interesting question in itself. However, insight into the form of the velocity profile within the boundary layer will perhaps provide a better understanding of the aforementioned efficiency increase.

In § 2 it is established that under certain conditions there exist similarity solutions of the boundary-layer equations for swirling flow through a convergent funnel whose boundary is the surface of revolution formed by the rotation of the curve $r_0(x)$, x being defined as the distance along the funnel wall. The governing equations are shown to depend on two parameters β_1, γ_1 involving $r_0(x)$, $U(x)$—the free stream velocity , and $\delta(x)$—the boundary-layer thickness. The conditions which these functions must satisfy to justify the similarity assumption are examined and equations derived which prescribe the form of the radial velocity $U(x)$ for which the similarity solutions are valid for general values of (β_1, γ_1). However, the conditions established in §2 do not suffice to ensure the existence or uniqueness of a corresponding boundary-layer flow and thus §3 is devoted to reassessing the conclusions of previous authors on systems of equations of a like nature and relating them to the governing equations in this case. In particular, a more physical criterion is suggested for selecting the appropriate solution when $\beta < 0$ in the Falkner–Skan equations. This section concludes with some tabulated solutions of the equations in which acceptable super-velocity profiles are demonstrated for certain values of β_1, γ_1.

In §§4, 5 an approximate method of solution is outlined for general funnel shapes whose geometry is incompatible with the similarity assumption. In particular the case of the perfect cone is considered and displacement thickness estimates obtained by the method are favourably compared with the work of previous authors. The assumption that the velocity profiles at any point along the cone may be related to a particular pair of β_1, γ_1 enables a comparison of profile estimates to be made. In §6 the relationship is outlined and favourable profile comparisons are demonstrated including super-velocity comparisons. A discussion of the results concludes the paper in §7.

2. The problem

A steady, laminar and rotationally symmetric flow is generated in a funnel whose boundary is the surface of revolution formed by the rotation of the curve $r = r_0(x)$ where the x co-ordinate is defined as the distance along the funnel wall and the co-ordinate at right angles to the wall is denoted by z. The flow can be considered as an inviscid flow with a thin boundary layer on the wall of the funnel. The free stream comprises a radial component of velocity, such that at the edge of the boundary layer the velocity is given by $U(x)$, together with a superimposed swirl velocity $A/r_0(x)$. On the assumption that the boundary-layer thickness $\delta \ll r_0(x)$ we have the appropriate boundary-layer equations as

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}-\frac{v^2}{r_0}\frac{dr_0}{dx} = U\frac{dU}{dx}-\frac{A^2}{r_0^2}\frac{r_0'}{r_0}+\nu\frac{\partial^2 u}{\partial z^2}, \tag{1}$$

$$u\frac{\partial v}{\partial x}+w\frac{\partial v}{\partial z}+\frac{uv}{r_0}\frac{dr_0}{dx} = \nu\frac{\partial^2 v}{\partial z^2}, \tag{2}$$

$$\frac{\partial}{\partial x}(r_0 u)+\frac{\partial}{\partial z}(r_0 w) = 0, \tag{3}$$

where in (1) we have used the assumption that the boundary layer is thin and that variation in the pressure, p, through its thickness may be neglected.

The boundary conditions governing (1), (2) and (3) are

$$\left.\begin{array}{ll}\text{(i)} & u = v = 0 \quad \text{on} \quad z = 0,\\ \text{(ii)} & u \to U(x) \quad \text{and} \quad v \to A/r_0(x) \quad \text{as} \quad z \to \infty,\\ \text{(iii)} & \text{No flow across the wall.}\end{array}\right\} \tag{4}$$

After noting from (3) that there exists a stream function $\psi(x, z)$ such that

$$r_0 u = \frac{\partial \psi}{\partial z}, \quad r_0 w = -\frac{\partial \psi}{\partial x}, \tag{5}$$

the variables are transformed by the substitutions

$$\left.\begin{array}{ll} u = U(x)\dfrac{\partial f}{\partial \eta}(\xi, \eta), & v = \dfrac{A}{r_0(x)} g(\xi, \eta), \\ \xi = \xi(x), & \eta = \dfrac{z}{\delta(x)}, \end{array}\right\} \tag{6}$$

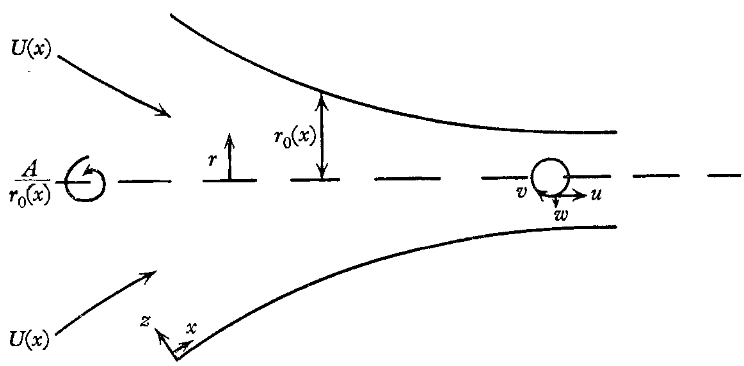

FIGURE 1

where $\delta(x)$ is the boundary-layer thickness. In terms of these substitutions we have that

$$\psi = \int r_0 u \, dz = r_0 U \delta f(\xi, \eta), \tag{7}$$

so that with the appropriate substitutions and the similarity assumption the momentum equations reduce to

$$f''' + \alpha f f'' + \beta(1 - f'^2) + \gamma(1 - g^2) = 0, \tag{8}$$

$$g'' + \alpha f g' = 0, \tag{9}$$

with boundary conditions

$$f'(\infty) = 1, \quad g(\infty) = 1, \quad f(0) = f'(0) = g(0) = 0, \tag{10}$$

where

$$\alpha = \frac{\delta(r_0 U \delta)'}{\nu r_0}, \quad \beta = \frac{\delta^2 U'}{\nu}, \quad \gamma = -\frac{A^2 \delta^2 r_0'}{\nu U r_0^3}, \tag{11}$$

are constants if (8) and (9) are to be solved with the boundary conditions (10).

The similarity solution can only exist if the functions $r_0(x)$, $\delta(x)$, $U(x)$ satisfy certain conditions which we now investigate. From (11) we have

$$2\gamma UU' = -\frac{2A^2\delta^2 r_0' U'}{\nu r_0^3} = -\frac{2A^2 r_0'}{\nu r_0^3}\beta, \tag{12}$$

and integrating gives

$$\gamma U^2 = \beta A^2\left(\frac{1}{r_0^2}-\frac{1}{a^2}\right), \tag{13}$$

where a is a constant of integration.

This makes $r_0 = a$ where $U = 0$ but a^2 can be replaced by $-a^2$ in the following argument to give a case where $U > 0$ throughout

$$\frac{\delta(r_0\delta)\,U'}{\nu r_0}+\frac{\delta U(r_0\delta)'}{\nu r_0} = \alpha, \tag{14}$$

whence

$$\frac{(r_0\delta)'}{r_0\delta} = \left(\frac{\alpha}{\beta}-1\right)\frac{U'}{U}, \tag{15}$$

$$(r_0\delta) = \text{constant}\; U^{(\alpha-\beta)/\beta} \quad (U > 0, \beta = 0). \tag{16}$$

Introducing a non-dimensional constant k, since

$$\frac{\delta}{a} \propto \left(\frac{\nu}{A}\right)^{\frac{1}{2}}, \tag{17}$$

$$(r_0\delta) = ka(\nu A)^{\frac{1}{2}}\left(\frac{a}{A}\right)^{\alpha/\beta}|U|^{(\alpha-\beta)/\beta} \tag{18}$$

and finally

$$\beta = \frac{\delta^2 U'}{\nu} = (r_0\delta)^2\frac{1}{r_0^2}\frac{U'}{\nu}, \tag{19}$$

so that substituting (13) and (18) into (19) gives

$$\beta = k^2(\nu A)\,a^2\left(\frac{a}{A}\right)^{2\alpha/\beta}|U|^{[2(\alpha-\beta)]/\beta}\left(\frac{1}{a^2}+\frac{\gamma U^2}{\beta A^2}\right)\frac{U'}{\nu}, \tag{20}$$

whence

$$\frac{\beta(x-x_0)}{k^2a} = \left\{\frac{\beta}{2\alpha-\beta}\left(\frac{a|U|}{A}\right)^{(2\alpha-\beta)/\beta}+\frac{\gamma}{2\alpha+\beta}\left(\frac{a|U|}{A}\right)^{(2\alpha+\beta)/\beta}\right\}, \tag{21}$$

so long as $\beta \neq \pm 2\alpha$, where x_0 is a constant of integration.

If $\beta = 2\alpha$ we have

$$\frac{\beta(x-x_0)}{k^2a} = \log\left(\frac{aU}{A}\right)+\frac{\gamma}{2\beta}\left(\frac{aU}{A}\right)^2, \tag{22}$$

and if $\beta = -2\alpha$

$$\frac{\beta(x-x_0)}{k^2a} = -\frac{1}{2}\left(\frac{aU}{A}\right)^{-2}+\frac{\gamma}{\beta}\log\left(\frac{aU}{A}\right). \tag{23}$$

The equations (21), (22) and (23) prescribe the form of the radial velocity $U(x)$ for which the similarity solutions are valid for general values of the constants α, β, γ. Equation (13) then gives the shape of the channel $r_0(x)$ and (16) in turn gives the form of the boundary-layer thickness $\delta(x)$.

The case $\beta = 0 \Rightarrow U = \text{constant} = U_0$ say and so

$$\frac{\delta U_0(r_0\delta)'}{\nu r_0} = \alpha. \tag{24}$$

Thus
$$\frac{\gamma(r_0\delta)'}{(r_0\delta)} = -\alpha\frac{A^2 r_0'}{U_0^2 r_0^2}, \tag{25}$$

and if $\gamma \neq 0$ then
$$(r_0\delta) = \text{const}\exp\left(\frac{\alpha A^2}{2\gamma U_0^2 r_0^2}\right) \tag{26}$$

and again introducing a non-dimensional constant k we have

$$(r_0\delta) = k\left(\frac{A}{U_0}\right)^2\left(\frac{\nu}{A}\right)^{\frac{1}{2}}\exp\left(\frac{\alpha A^2}{2\gamma U_0^2 r_0^2}\right), \tag{27}$$

and hence
$$\gamma = -\frac{A^2 r_0'}{\nu U_0 r_0^5}k^2\left(\frac{A}{U_0}\right)^4\left(\frac{\nu}{A}\right)\exp\left(\frac{\alpha A^2}{\gamma U_0^2 r_0^2}\right), \tag{28}$$

so that if $\alpha \neq 0$
$$\frac{2\alpha^2 U_0(x-x_0)}{\gamma A k^2} = \left(\frac{\alpha A^2}{\gamma U_0^2 r_0^2} - 1\right)\exp\left(\frac{\alpha A^2}{\gamma U_0^2 r_0^2}\right), \tag{29}$$

which includes three arbitrary constants U_0, k, x_0.

The case $\alpha = 0$ implies
$$\gamma = -\frac{k^2 A^5 r_0'}{U_0^5 r_0^5}, \tag{30}$$

so that
$$\frac{4\gamma U_0(x-x_0)}{Ak^2} = \left(\frac{A}{Ur_0}\right)^4, \tag{31}$$

i.e.
$$\frac{U_0 r_0}{A} = \left(\frac{Ak^2}{4\gamma U_0}\right)^{\frac{1}{4}}(x-x_0)^{-\frac{1}{4}}. \tag{32}$$

This last case, however, is seen to be impossible since it implies that $g'' = 0$ for all η so that $g = a + b\eta$, i.e. a form which cannot satisfy the boundary conditions $g(0) = 0$, $g(\infty) = 1$.

Finally we note that the equations for f and g contain three parameters α, β, γ, but these can be reduced effectively to two since $\delta(x)$ is indeterminate to the extent of a multiplicative factor. Thus $\delta(x)$ can be scaled arbitrarily with corresponding scaling of f and η, i.e. $\eta = \lambda\eta_1 \Leftrightarrow \delta_1 = \lambda\delta$ and $f(\eta) = \lambda f_1(\eta_1)$. Hence, the parameters can be modified to $\alpha_1 = \lambda^2\alpha$, $\beta_1 = \lambda^2\beta$, $\gamma_1 = \lambda^2\gamma$, where λ must be positive. Now αf must be > 0 since at infinity $g'' < 0$ and $g' > 0$, but also $f > 0$ for large η so that $\alpha > 0$ allows us to choose $\lambda^2 = 1/\alpha$, from which we have

$$\alpha_1 = 1, \quad \beta_1 = \beta/\alpha \quad \text{and} \quad \gamma_1 = \gamma/\alpha.$$

The problem thus reduces to the solution of the following equations:

$$f''' + ff'' + \beta_1(1-f'^2) + \gamma_1(1-g^2) = 0, \tag{33}$$

$$g'' + fg' = 0, \tag{34}$$

where
$$f(0) = f'(0) = g(0) = 0; f'(\infty) = g(\infty) = 1. \tag{35}$$

3. The existence and uniqueness of solutions of the similarity equations

Although a set of necessary conditions for similar velocity profiles has been obtained in the previous section, the solution of the boundary-layer equations will exist only if (33) and (34) can actually be solved with the boundary conditions (35). The conditions (11) do not suffice to ensure the existence or uniqueness of a corresponding boundary-layer flow. Such questions demand a study of the differential equations (33) and (34). However, we note that if $\gamma_1 = 0$, (33) reduces directly to the Falkner–Skan equation which both Hartree (1937) and Stewartson (1954) have discussed as regards existence and uniqueness of solutions.

Hartree (1937) obtained useful information by considering the possible behaviour of the solutions for large η. He showed that if $q(\eta) = 1-f(\eta)$ then $q(\eta)$ could be approximated to in terms of the asymptotic expansions of the parabolic cylinder functions and in fact as $\eta \to \infty$

$$q(\eta) \sim A \exp\{-\tfrac{1}{2}\xi^2\}\xi^{-(2\beta+\alpha)/\alpha} + B\xi^{2\beta/\alpha} \quad (\alpha > 0), \tag{36}$$

where $$\xi = (\eta - \Delta_1)|\alpha|^{\frac{1}{2}} \quad \text{and} \quad \Delta_1 = \int_0^\infty (1-f')\,d\eta.$$

Thus, if $\beta \geqslant 0$ the condition $q(\infty) = 0$ requires $B = 0$ and then $q \to 0$ with exponential rapidity. On the other hand, if β is negative any expression of the form (36) $\to 0$ as $\eta \to \infty$ so that a range of values of $f''(0)$ may give a solution satisfying the condition at infinity, i.e. there is a whole family of integrals of (33) ($\gamma = 0$) satisfying the appropriate boundary conditions. To make the solutions for $\beta < 0$ unique and appropriate to the application to be made of them the condition at infinity was replaced by (*a*) $f' \to 1$ from below as $\eta \to \infty$, (*b*) $f' \to 1$ as rapidly as possible (with $f' \leqslant 1$).

In formulating (*a*) Hartree suggested that $f' \to 1$ from above would imply a reversal of the normal gradient of the tangential velocity in the boundary layer which would lead to a solution unlikely to be physically significant, while to endorse (*b*) he used continuity arguments. On (*b*) Stewartson commented "such arguments are not altogether satisfactory, because it is easy to produce solutions, different from Hartree's but satisfying the continuity arguments, though (*b*) seems to give the analytic continuation of the solutions for $\beta > 0$". He went on to propose another condition at infinity which he hoped would be more convincing; however his new criterion yielded exactly Hartree's solutions for the range $\beta_0 < \beta < 0 (f''(0) > 0)$. Here β_0 is the lower limit of significance, i.e. where $f''(0) = 0$, and was found by Hartree to be $-0{\cdot}1988$.

There are then various comments pertaining to the problem on hand which arise out of the preceding discussion. The first concerns the necessity for a physical criterion which allows for the occurrence of super-velocities within the boundary layer, where by 'super-velocity' we imply that the value of a velocity component within the boundary layer exceeds its corresponding free stream value. If we question the reason Hartree gives for his condition (*a*) and suggest that the criterion to be satisfied is that the resultant velocity within the boundary shall not exceed the total free stream velocity for reasons of pressure balance, i.e. that the total head within the boundary layer shall not exceed the total head on streamlines external to the boundary layer, we see that Hartree's condition (a) remains correct for the cases he considered while falling within the framework of the above suggestion. Moreover, we now have a criterion which allows the occurrence of super-velocities to be physically tenable.

We can also obtain useful information by considering the behaviour of solutions for large values of η. As $\eta \to \infty, f'$ and $g \to 1$ and again

$$\eta - f \to \Delta_1 = \int_0^\infty (1-f')\,d\eta,$$

which must be finite if the displacement thickness of the boundary layer is finite. Hence, when η is large, the equations for $q(\eta) = 1-f'(\eta)$ and $p(\eta) = 1-g(\eta)$ approximate to

$$q'' + \alpha(\eta-\Delta_1)q' - 2\beta q - 2\gamma p = 0, \tag{37}$$

$$p'' + \alpha(\eta-\Delta_1)p' = 0. \tag{38}$$

Equation (38) can be integrated directly to give

$$p = C\int_\xi^\infty \exp\{-\tfrac{1}{2}t^2\}\,dt, \tag{39}$$

where again

$$\xi = (\eta-\Delta_1)\,|\alpha|^{\frac{1}{2}}.$$

When $\alpha > 0$ and as $\eta \to +\infty$ the general solution of the resultant equation

$$q'' + \alpha(\eta-\Delta_1)q' - 2\beta q = 2\gamma C\int_\xi^\infty \exp\{-\tfrac{1}{2}t^2\}\,dt \tag{40}$$

is

$$q(\eta) \sim A\exp\{-\tfrac{1}{2}\xi^2\}\xi^{-(2\beta+\alpha)/\alpha} + B\xi^{2\beta/\alpha} - C\frac{\gamma}{\beta}\int_\xi^\infty \exp\{-\tfrac{1}{2}t^2\}\,dt. \tag{41}$$

Since the term extra to (36) $\to 0$ as $\eta \to \infty$ irrespective of the sign of β the discourse on the existence of solutions of the Falkner–Skan equations will apply equally here, so that we shall seek solutions for f' and g which $\to 1$ as rapidly as possible within the limits of the physical criterion suggested.

Finally, we refer to the equation of similar profiles for flow with suction, namely

$$f''' + \alpha ff'' + \beta(1-f'^2) + \gamma f'' = 0, \tag{42}$$

where, as in the problem on hand, two parameters are present in effect. For (42), Iglisch & Kemnitz (1955) have shown that in the case $\alpha = 1$, $\beta < 0$ solutions subject to $f(0) = f'(0) = 0, f'(\infty) = 1$ and $0 < f' < 1$ in $0 < \eta < \infty$ exist only for $\gamma \geqslant \gamma_0(\beta)$ where $\gamma_0(\beta)$ is a function which increases with $|\beta|$. When $\gamma = \gamma_0(\beta)$ there is only one such profile which has $f''(0) = 0$ and so gives a separation profile. When $\gamma > \gamma_0(\beta)$ there is a solution in which $f' \to 1$ more rapidly than any other, thus satisfying Hartree's condition. We might expect, therefore, the existence of a similar function $\gamma_0^*(\beta_1)$ say, such that for $\gamma_1 = \gamma_0^*(\beta_1), f''(0) = 0$ and that the solution in this case be unique. Again $\gamma_0^*(\beta_1)$ will divide the (γ_1, β_1)-plane into two regions, one of no solutions and one of many solutions, in the latter of which the appropriate solution is again that which satisfies the physical criterion suggested together with Hartree's criterion (*b*) that $f' \to 1$ as rapidly as possible.

In conclusion we note that (33) and (34) have a trivial solution when $\beta + \gamma = 0$, namely $f' = g =$ the Blasius profile.

Numerical solutions of the equations of similar profiles have been obtained for various (β_1, γ_1) and are included in tables 1–5 (obtainable from the Editor). These solutions demonstrate the occurrence of super-velocities within the boundary layer for particular values of β_1 and γ_1. The separation profile solutions for $f''(0) = 0$ have facilitated the plotting of the anticipated function $\gamma_0^*(\beta_1)$ for small γ_1 (figure 2).

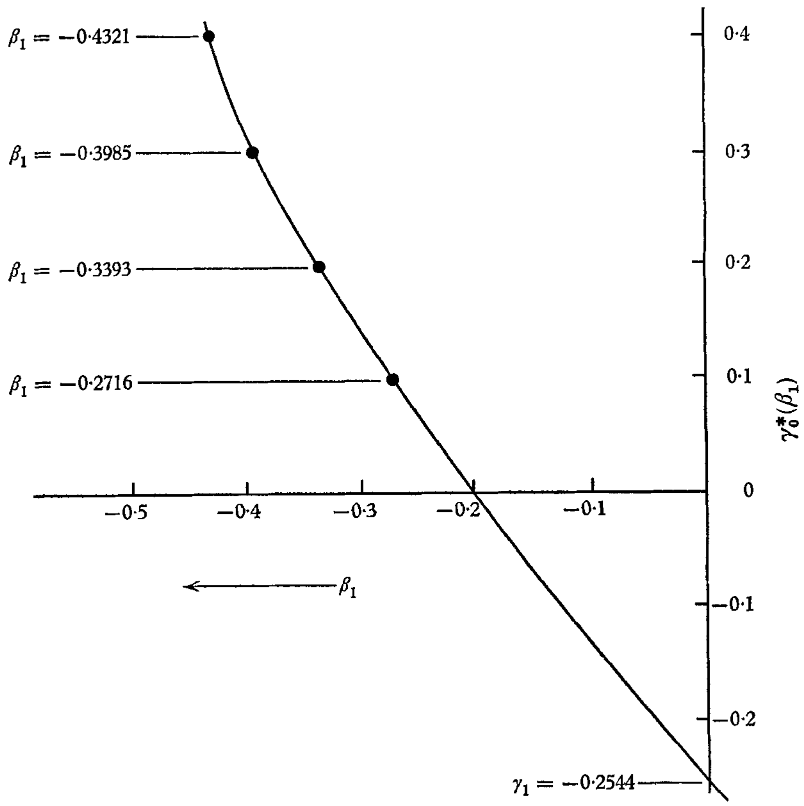

FIGURE 2. The function $\gamma_0^*(\beta_1)$.

4. The perfect cone

The particular case of a perfectly conical funnel clearly falls within the framework of a 'convergent funnel of arbitrary shape' but it is not one in which the assumption of similar profiles will hold, i.e. we cannot expect the velocity profiles within the boundary layer to be independent of distance along the cone. However, we can anticipate that the profiles at any point will be related to a particular pair of (β_1, γ_1). To establish a comparison of profiles in this way and to develop an alternative method of solution to the lengthy iterative procedure of Garbsch (1956) we investigate the problem posed in the title by approximate techniques. A solution is reached using the integrated form of the boundary-layer equations and extending the Wieghardt (1946) two-parameter method of two-dimensional flow. Various authors have made rather drastic assumptions in this type of work in two-dimensional flows with surprising success. An assumption fundamental to the solution that follows is that the relationships between the momentum thickness and boundary-layer thickness and between the mixed momentum thickness and boundary-layer thickness are basically linear. This assumption enables a considerable simplification to be made in the process of solving the momentum integral equations. The two parameters for which these equations are then solved allow various characteristic properties of the boundary layer and of the velocity profile to be determined as functions of distance along the cone.

We consider the steady laminar flow of liquid through a perfect cone whose semi-angle is θ. The solution that follows applies equally well to an arbitrary channel shape $r_0(x)$ and it is not until later that the specifications of the particular shape we have chosen are required. The free stream flow is considered to have both radial and swirling components of velocity, the swirling component being represented as a potential line vortex placed on the axis of the funnel.

On the same assumptions outlined in §2 the equations of motion governing the flow within the boundary layer will again be (1), (2) and (3). Integrating (1) and (2) across the boundary layer and using (3) gives the u-momentum integral equation as

$$\frac{1}{r_0}\frac{d}{dx}\left(r_0\int_0^\infty u(u-U)\,dz\right)+\frac{dU}{dx}\int_0^\infty (u-U)\,dz + \frac{r_0'}{r_0}\int_0^\infty (V^2-v^2)\,dz = -\nu\left(\frac{\partial u}{\partial z}\right)_0, \tag{43}$$

and

$$\frac{1}{r_0^2}\frac{d}{dx}\left(r_0^2\int_0^\infty u(u-V)\,dz\right) = -\nu\left(\frac{\partial v}{\partial z}\right)_0 \tag{44}$$

as the v-momentum integral equation which, together with (43), we require to solve subject to the boundary conditions

$$\left.\begin{array}{ll}
\left.\begin{array}{ll}
\text{(i)} & u = 0 \\
\text{(ii)} & -\nu\dfrac{\partial^2 u}{\partial z^2} = U\dfrac{dU}{dx} - V^2\dfrac{r_0'}{r_0}
\end{array}\right\} & \text{on} \quad z = 0, \\
\left.\begin{array}{ll}
\text{(iii)} & u = U(x) \\
\text{(iv)} & \dfrac{\partial u}{\partial z} = 0 \\
\text{(v)} & \dfrac{\partial^2 u}{\partial z^2} = 0
\end{array}\right\} & \text{when} \quad z = \infty,
\end{array}\right\} \tag{45}$$

and

$$\left.\begin{array}{lll} \text{(i)} & v = 0 & \\ \text{(ii)} & \dfrac{\partial^2 v}{\partial z^2} = 0 & \text{on} \quad z = 0, \\ \text{(iii)} & \dfrac{\partial v}{\partial z} = 0 & \\ \text{(iv)} & v = V(x) = \dfrac{A}{r_0(x)} & \text{when} \quad z = \infty. \\ \text{(v)} & \dfrac{\partial^2 v}{\partial z^2} = 0 & \end{array}\right\} \tag{46}$$

As is usual when dealing with the integrated forms of the boundary-layer equations the method of solution depends upon the substitution in the momentum integral equations of a prescribed velocity profile which satisfies *some of the* boundary conditions. It is hoped that this will approximate to the exact profile which satisfies all the conditions as well as the momentum integral equation. If we consider an analogy with Pohlhausen's method, setting

$$u/U = f(\eta, \lambda_1), \quad v/V = g(\eta, \lambda_2) \quad \text{and} \quad \eta = \frac{z}{\delta(x)}.$$

would seem appropriate where the prescribed f and g have an x dependence through coefficients λ_1 and λ_2 chosen to satisfy certain of the boundary conditions. Unfortunately, however, such a direct extension of Pohlhausen's method is impossible, since the new form of the boundary conditions gives

$$g''(0) = g'''(0) = 0$$

thus preventing the inclusion of an analogous coefficient λ_2 in profile g. Consequently we must decide upon an alternative parameter to the second derivative of g or even seek two new parameters such as the two component boundary-layer thicknesses δ_x, δ_z. In view of the suggestion of Garbsch's work that there is little change in the g profile and since it is the f profile which holds most interest, it was decided that the second parameter should be incorporated in the f profile while a fixed g profile be established from the boundary conditions on v so that

$$u/U = f(\eta, \lambda_1, \lambda_2); \quad v/V = g(\eta).$$

The boundary conditions, after setting $\xi = x/c$, where c is the length of the cone along a generator, now become

$$\left.\begin{array}{lll} \text{(i)} & f = 0 & \\ \text{(ii)} & -\dfrac{\nu U}{\delta^2} f''(0) = \dfrac{U}{c}\dfrac{du}{d\xi} - \dfrac{V^2}{c r_0}\dfrac{dr_0}{d\xi} & \text{on} \quad \eta = 0, \\ \text{(iii)} & f = 1 & \\ \text{(iv)} & f' = 0 & \text{on} \quad \eta = 1, \\ \text{(v)} & f'' = 0 & \end{array}\right\} \tag{47}$$

and

$$\left.\begin{array}{lll} \text{(i)} & g = 0 & \multirow{2}{*}{on $\eta = 0$,} \\ \text{(ii)} & g'' = 0 & \\ \text{(iii)} & g = 1 & \\ \text{(iv)} & g' = 0 & \text{on } \eta = 1. \\ \text{(v)} & g'' = 0 & \end{array}\right\} \tag{48}$$

Although strictly the conditions at infinity are only approached asymptotically, it is assumed that those conditions can be transferred from infinity to $z = \delta$ (i.e. $\eta = 1$) without appreciable error.

The profile obtained for g after applying the five boundary conditions (48) to an appropriate polynomial is

$$g(\eta) = 2\eta - 2\eta^3 + \eta^4. \tag{49}$$

The profile for $f(\eta) = f(\eta, \lambda_1, \lambda_2)$. One of the most accurate approximate methods of two-dimensional boundary-layer theory is that whereby the momentum integral and energy integral equations are solved simultaneously. The particular method adopted by Wieghardt involved the use of a special two-parameter class of profiles which fulfilled exactly the boundary conditions (47). The main disadvantage of this doubly-infinite family of profiles is that it becomes inadequate in a region of sharply falling pressure where profiles with $f > 1$ are encountered which are normally physically untenable. However, in this problem this disadvantage is turned to advantage in that we require a family of profiles which in fact allow for the occurrence of super velocities in a region of falling pressure gradient.

Consequently we set

$$u/U = f(\eta, \lambda_1, \lambda_2) = f_1(\eta) + \lambda_1 f_2(\eta) + \lambda_2 f_3(\eta), \tag{50}$$

where

$$\left.\begin{aligned} f_1(\eta) &= 1 - (1-\eta)^8(1 + 8\eta + 36\eta^2 + 120\eta^3), \\ f_2(\eta) &= (1-\eta)^8\eta(1 + 8\eta + 36\eta^2), \\ f_3(\eta) &= -(1-\eta)^8\eta^2(1 + 8\eta), \end{aligned}\right\} \tag{51}$$

and

$$\left.\begin{aligned} &f_1(0) = 0; \quad f_1'(0) = 0; \quad f_1''(0) = 0, \\ &f_2(0) = 0; \quad f_2'(0) = 1; \quad f_2''(0) = 0, \\ &f_3(0) = 0; \quad f_3'(0) = 0; \quad f_3''(0) = -2, \end{aligned}\right\} \tag{52}$$

so that $f_1(\eta)$ provides a basic profile to which varying amounts of the function $f_2(\eta)$ and $f_3(\eta)$ may be added independently to modify the slope and curvature respectively at the surface. Relating λ_1 and λ_2 to the boundary conditions, we have

$$\text{(i)} \quad \mu\left(\frac{\partial u}{\partial z}\right)_0 = \frac{\mu U}{\delta}\left(\frac{\partial f}{\partial \eta}\right)_0 = \frac{\mu U}{\delta} f_2'(0)\lambda_1 = \frac{\mu U \lambda_1}{\delta}, \tag{53}$$

so that

$$\frac{\tau_u}{\rho U^2} = \frac{\nu}{U\delta}\lambda_1, \quad \text{i.e.} \quad \lambda_1 = \frac{\tau_u \delta}{\mu U}, \tag{54}$$

where τ_u is the x component of the skin friction at the wall;

$$\text{(ii)} \quad -\nu\left(\frac{\partial^2 u}{\partial z^2}\right)_0 = U\frac{dU}{dx}-\frac{V^2}{r_0}\frac{dr_0}{dx} = \frac{U}{c}\frac{dU}{d\xi}-\frac{V^2}{cr_0}\frac{dr_0}{d\xi}, \tag{55}$$

which becomes

$$U\lambda_2 = \frac{U\delta^2}{2\nu c}\left\{\frac{dU}{d\xi}-\frac{V^2}{Ur_0}\frac{dr_0}{d\xi}\right\}. \tag{56}$$

5. The equations

With the non-dimensional variables $\eta = z/\delta$ and $\xi = x/c$ together with the substitutions $u = Uf(\eta)$ and $v = (A/r_0)g(\eta)$, equations (43) and (44) become

$$\frac{1}{cr_0}\frac{d}{d\xi}\left(r_0 U^2\delta\int_0^1 f(f-1)\,d\eta\right)+\frac{1}{c}\frac{dU}{d\xi}U\delta\int_0^1(f-1)\,d\eta$$
$$+\frac{1}{cr_0}\frac{A^2}{r_0^2}\frac{dr_0}{d\xi}\delta\int_0^1(1-g^2)\,d\eta = -\nu\frac{U}{\delta}f'(0) \tag{57}$$

and

$$\frac{1}{cr_0^2}\frac{d}{d\xi}\left(r_0^2 U\frac{A}{r_0}\delta\int_0^1 f(1-g)\,d\eta\right) = \frac{\nu A}{\delta r_0}g'(0). \tag{58}$$

If we now assume a basically linear relationship between the momentum thickness

$$\left(\delta\int_0^1 f(1-f)\,d\eta\right)$$

and the boundary-layer thickness δ and between the mixed momentum thickness

$$\left(\delta\int_0^1 f(1-g)\,d\eta\right)$$

and δ, the result is to simplify (57) and (58) to

$$\frac{r_0'}{r_0}+\frac{U'}{U}(2+A_1)+\frac{\delta'}{\delta} = \frac{\nu c}{U\delta^2}A_2+\frac{r_0'}{r_0}\frac{A^2}{r_0^2U^2}A_3, \tag{59}$$

where

$$A_1 = \frac{\int_0^1(1-f)\,d\eta}{\int_0^1 f(1-f)\,d\eta} = \frac{\delta_{1x}}{\delta_{2x}} \quad (=H) \tag{60}$$

and δ_{1x} = displacement thickness in the x direction,

δ_{2x} = momentum thickness in the x direction,

$$A_2 = \frac{f'(0)}{\int_0^1 f(1-f)\,d\eta} = \frac{\lambda_1}{\int_0^1 f(1-f)\,d\eta}, \tag{61}$$

$$A_3 = \frac{\int_0^1(1-g^2)\,d\eta}{\int_0^1 f(1-f)\,d\eta}, \tag{62}$$

and
$$\frac{r_0'}{r_0}+\frac{U'}{U}+\frac{\delta'}{\delta} = \frac{\nu c}{U\delta^2}A_4, \tag{63}$$

where
$$A_4 = \frac{g'(0)}{\int_0^1 f(1-g)\,d\eta} = \frac{2}{\int_0^1 f(1-g)\,d\eta}. \tag{64}$$

The profile dependence on x (equivalently ξ) is now re-adopted through solutions for λ_1, λ_2 which provide a form of perturbation on the assumed linearity.

Introducing a new parameter $G = U\delta^2/\nu c$, equations (59) and (63) become

$$\frac{r_0'}{r_0}+\frac{U'}{U}(2+A_1)+\frac{\delta'}{\delta} = \frac{A_2}{G}+\frac{r_0'}{r_0}\frac{A^2}{r_0^2U^2}A^2 \tag{65}$$

and
$$\frac{r_0'}{r_0}+\frac{U'}{U}+\frac{\delta'}{\delta} = \frac{A_4}{G}. \tag{66}$$

Now
$$\frac{G'}{G} = \frac{2\delta'}{\delta}+\frac{U'}{U}$$

which from (66) can be rewritten

$$\frac{G'}{G} = \frac{2A_4}{G}-\frac{2r_0'}{r_0}-\frac{U'}{U}. \tag{67}$$

Deriving an expression for U'/U from (65) and (66) enables us to write

$$\frac{G'}{G} = \frac{2A_4}{G}-\frac{2r_0'}{r_0}-\frac{1}{1+A_1}\left\{\frac{A_2-A_4}{G}+\frac{r_0'}{r_0}\frac{A^2}{r_0^2U^2}A_3\right\}, \tag{68}$$

so that finally

$$G' = 2A_4-\frac{1}{1+A_1}(A_2-A_4)-G\frac{r_0'}{r_0}\left\{2+\frac{A^2}{r_0^2U^2}\frac{A_3}{(1+A_1)}\right\} \tag{69}$$

and
$$A_2 = G\frac{U'}{U}(1+A_1)+A_4-G\frac{r_0'}{r_0}\frac{A^2}{r_0^2U^2}A_3, \tag{70}$$

while (56) now yields
$$U\lambda_2 = \frac{G}{2}\left\{U'-\frac{A^2}{r_0^2U^2}\frac{r_0'}{r_0}\right\}. \tag{71}$$

On the assumption that initially the boundary layer has zero thickness, we have the following boundary conditions

$$G(0) = 0; \quad \lambda_2(0) = 0 \tag{72}$$

in which case, we approach the required solution in the following way. Firstly, we insert (72) in (70) and solve a simple quadratic in $\lambda_1(0)$. In conjunction with (72) and (69) this value enables us to determine $G'(0)$. We are now in a position to proceed with a step-by-step integration of the equation by using the slope of G at a point ξ to give an approximation to G at $\xi+d\xi$.

It is here that specifications particular to the perfect cone are introduced. From continuity considerations the non-dimensional model of the radial velocity component at the edge of the boundary layer is represented by $\overline{U}(\xi) = 1/(1-\xi)^2$ while the swirling component of velocity is simply given by $\overline{V}(\xi) = K/(1-\xi)$ so that K gives a measure of the relative magnitudes of the swirl and radial components of velocity at the entrance to the cone.

With the free stream thus specified the equations have been solved for various values of K to obtain the displacement thickness for each K. The results are demonstrated in figures 3 and 4.

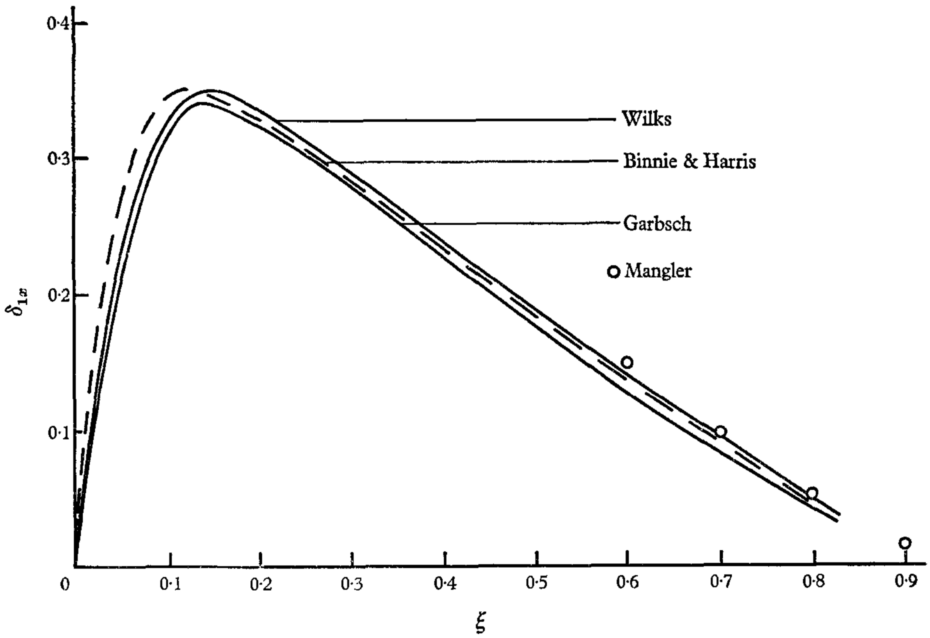

FIGURE 3. Displacement thickness when the swirl is zero ($K = 0$).

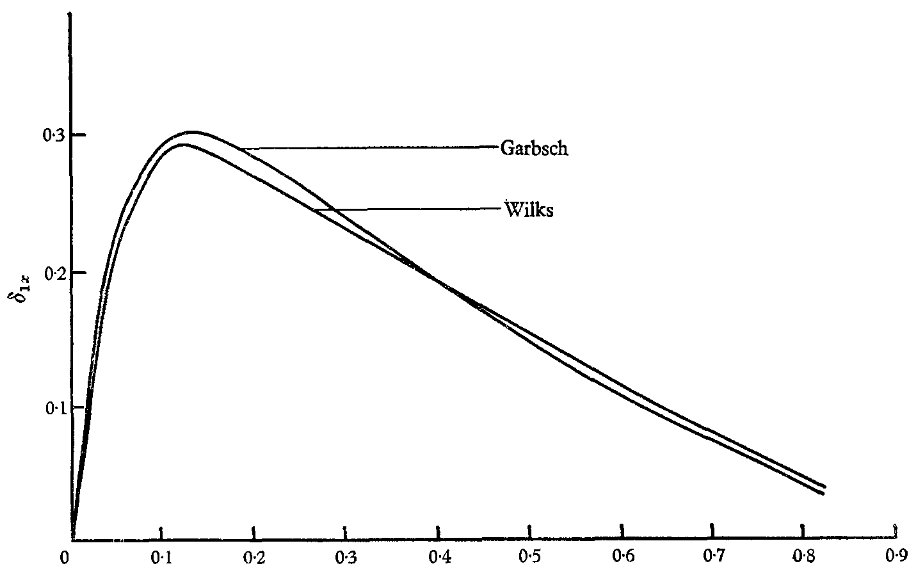

FIGURE 4. Displacement thickness when $K = 1$.

6. Comparison of profiles

It is clear that §§4 and 5 are related to the earlier sections of this work in that the perfect cone is simply a particular case of a 'convergent funnel of arbitrary shape'. However, as can be seen in §5 it is not a shape for which the velocity profile within the boundary layer remains 'similar'. Nevertheless, a comparison of results is possible since we anticipate that the profile at any particular stage of the cone wall will be connected with a particular pair of (β_1, γ_1) of the similarity solutions.

First, we note that from (5)

$$(r_0\delta)'/r_0\delta = ([\alpha/\beta]-1)(U'/U)$$

so that considering the specifications made in the cone flow solution we have

$$\frac{\delta'}{\delta}-\frac{1}{1-\xi}=\left(\frac{\alpha}{\beta}-1\right)\frac{2}{1-\xi}. \tag{73}$$

Clearly when $\delta' = 0$, i.e. at the point where the boundary layer thickness is greatest, we always have

$$\beta = 2\alpha, \quad \text{i.e.} \quad \beta_1 = 2. \tag{74}$$

Consequently, we can say that the similarity profile corresponding to that obtained by the approximate method at the point where the boundary thickness is at a maximum is one involving $\beta_1 = 2$, while γ_1 is determined from the initial specification of the swirl component of velocity. Immediately this suggests a comparison of profiles in the trivial case $(\gamma_1 = 0) \equiv (K = 0)$; $\beta_1 = 2$ for which the similarity profile is well known.

In general, however, a comparison of the two derived profiles at any point is possible, so long as we can determine the scaling factor $1/\alpha$.

By definition

$$\alpha = \frac{\delta(r_0 U\delta)'}{r_0 \nu c}, \tag{75}$$

$$= G\left(\frac{\delta'}{\delta}+\frac{r_0'}{r_0}+\frac{U'}{U}\right), \tag{76}$$

where again $G = U\delta^2/\nu c$, so that in fact, in the case of the cone from (66),

$$\alpha \equiv A_4. \tag{77}$$

We also have from (56)

$$2\lambda_2/\alpha = \beta_1+\gamma_1 \tag{78}$$

and finally, by definition and specification,

$$2\gamma_1/\beta_1 = K^2(1-\xi)^2. \tag{79}$$

Thus simple calculations involving (77), (78) and (79) enable the similarity values β_1, γ_1 to be determined corresponding to a given approximate method profile at any point along the cone. Various comparisons are demonstrated in figures 5–9.

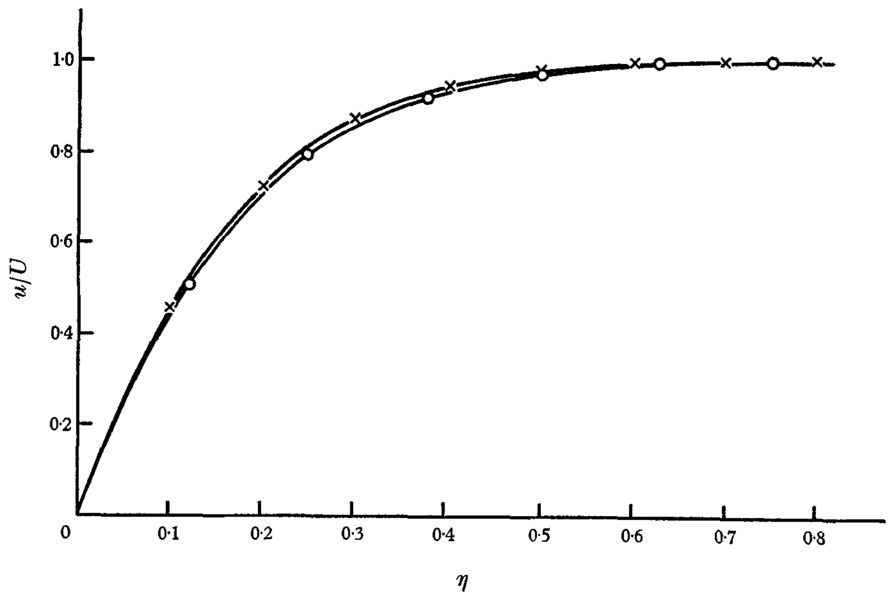

FIGURE 5. Profiles for zero swirl: ×, approximate solution; ○, similarity solution.

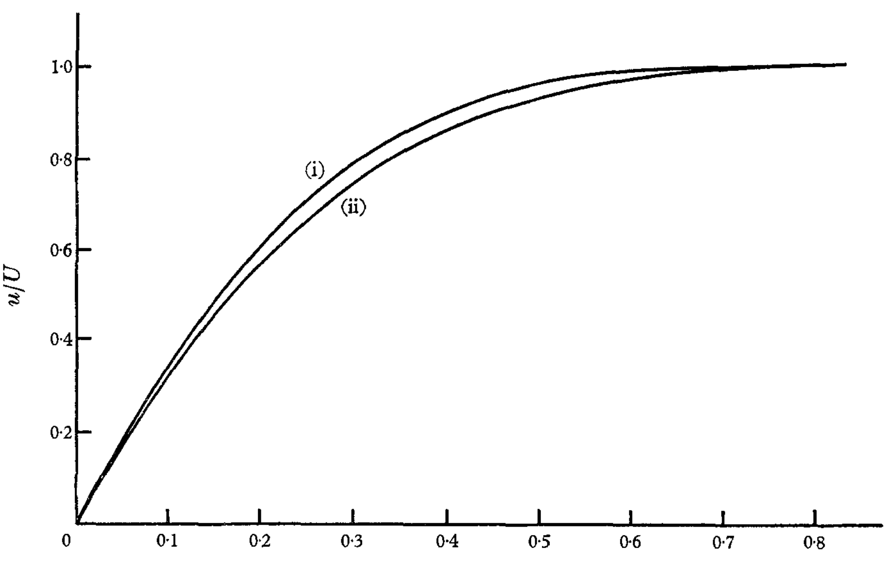

FIGURE 6. (i) Approximate solution $K = 1$, $\xi = 0{\cdot}1$. (ii) Similarity solution $\beta_1 = 0{\cdot}5102$, $\gamma_1 = 0{\cdot}2066$.

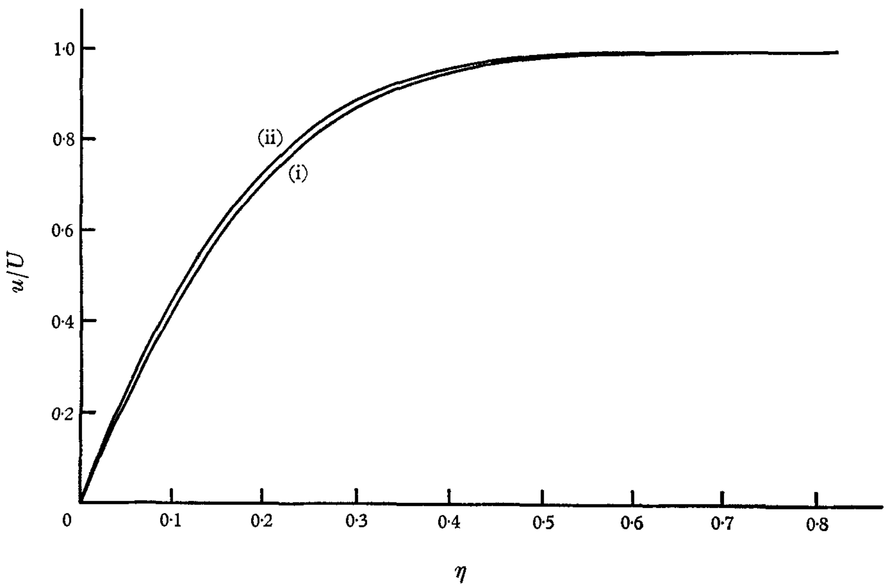

FIGURE 7. (i) Approximate solution $K = 1$, $\xi = 0{\cdot}2$. (ii) Similarity solution $\beta_1 = 1{\cdot}1960$, $\gamma_1 = 0{\cdot}3826$.

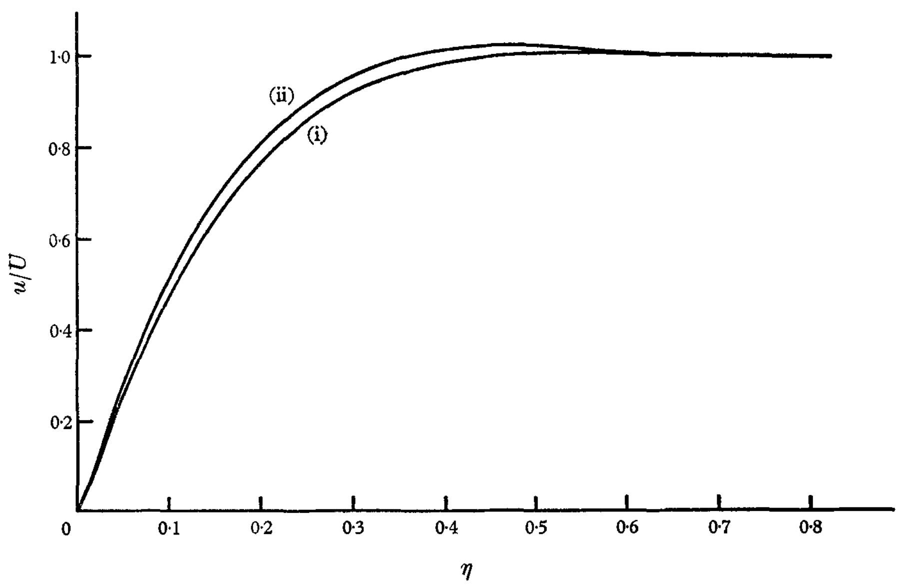

FIGURE 8. (i) Approximate solution $K = 2$, $\xi = 0{\cdot}1$. (ii) Similarity solution $\beta_1 = 0{\cdot}5445$, $\gamma_1 = 0{\cdot}8820$.

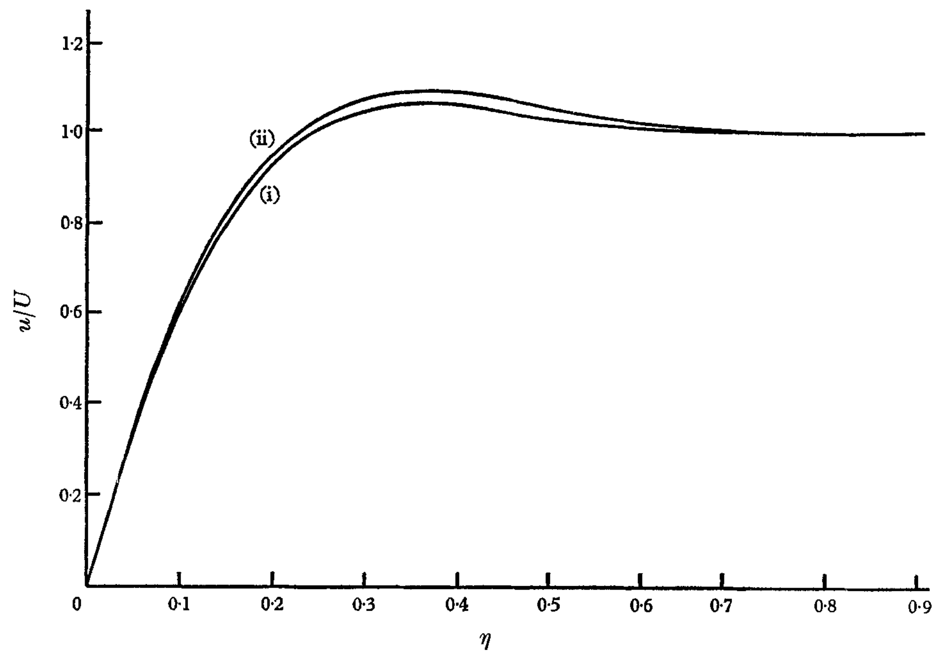

FIGURE 9. (i) Approximate solution $K = 2$, $\xi = 0{\cdot}2$. (ii) Similarity solution $\beta_1 = 1{\cdot}2400$, $\gamma_1 = 1{\cdot}5870$.

7. Discussion of results

It is important to keep the problem in physical perspective, particularly with reference to the required conditions under which the solution was derived, namely steady, laminar flow. We have demonstrated that when both radial and swirling components of velocity exist in the free stream then the phenomenon of super-velocity occurs, i.e. the pressure gradient (breakdown) contribution to the boundary-layer velocity in the exit direction due to swirl (as outlined by Taylor) is sufficient in some instances to enhance the normal reduction of this velocity component by friction to such an extent as to cause the velocity within the boundary layer to exceed the corresponding free stream component in that direction. The introduction of swirl into the flow thus tends always to increase the amount of fluid discharged and in fact results for larger values of K demonstrate negative displacement thicknesses, i.e. discharge efficiencies greater than the equivalent ideal flow. This is not as unacceptable as it appears at first sight since Taylor's pure swirl problem demonstrates finite radial velocities within the boundary layer despite the absence of a corresponding free stream component and hence provides a limit example of a discharge efficiency greater than that of its equivalent ideal flow.

It must be remembered that this discussion of efficiency relates to flows in which fluid occupies the entire funnel and does not allow for flows in which a vortex core has been established. Binnie & Hookings (1948) have examined the transition from one state to the other in terms of swirl magnitude and show that flows occur for only relatively small values of swirl in which the fluid occupies the whole funnel. Hence we can only conclude that the more swirl that can be introduced compatible with steady, laminar, core-free flow, the more efficient will be the discharge compared with the corresponding radial flow.

Finally we note that agreement between the approximate method results and results of previous workers is good. Binnie & Harris (1950) remark that when the swirl component is zero, comparison may be made with Mangler's work on exact solutions of the axially symmetric boundary-layer equations. Although Mangler's original similarity solution for this case has been questioned in recent literature, Brown & Stewartson (1965) resolved the problem when they demonstrated that similarity solutions with algebraic decay can be limit solutions of the full boundary layer equations with exponential decay. Their work confirmed the validity of the assumption that the outer flow near the apex is potential sink flow, i.e. $U \propto r^{-2}$ and hence corroborated Mangler's (1948) solution. The fact then that in the approximate method herein the non-dimensional free stream radial component of velocity was taken to be of the form $\overline{U}(\xi) = (1-\xi)^{-2}$ would suggest that there is some agreement between the estimated displacement thickness in that region. Mangler in fact suggests the result that the displacement thickness near the apex of the cone is given by $0{\cdot}587\ (1-\xi)^{\frac{3}{2}}$ and it can be seen from the graph of displacement thickness for zero swirl how close is the agreement between this estimate and the approximate method results. That this is so is no doubt attributable to the highly favourable pressure gradient in this region. The comparison of velocity profiles provides a further confirmation of the usefulness of the approximation of § 5 when applied in favourable well-behaved circumstances.

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